ELSEVIER
Journal of Computational and Applied Mathematics 139 (2002) 141–161
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Planar G^2 Hermite interpolation with some fair, C-shaped curves

D.S. Meek *, D.J. Walton

Department of Computer Science, University of Manitoba, Winnipeg, Man., Canada R3T 2N2

Received 10 November 2000

Abstract

G^2 Hermite data consists of two points, two unit tangent vectors at those points, and two signed curvatures at those points. The planar G^2 Hermite interpolation problem is to find a planar curve matching planar G^2 Hermite data. In this paper, a C-shaped interpolating curve made of one or two spirals is sought. Such a curve is considered fair because it comprises a small number of spirals. The C-shaped curve used here is made by joining a circular arc and a conic in a G^2 manner. A curve of this type that matches given G^2 Hermite data can be found by solving a quadratic equation. The new curve is compared to the cubic Bézier curve and to a curve made from a G^2 join of a pair of quadratics. The new curve covers a much larger range of the G^2 Hermite data that can be matched by a C-shaped curve of one or two spirals than those curves cover.

MSC: 65S05

Keywords: Planar G^2 Hermite interpolation; Fair curves; C-shaped curves

1. Introduction

A set of G^2 Hermite data consists of two points, unit tangent vectors at those points, and signed curvatures at those points. The planar G^2 Hermite interpolation problem is to find a planar curve matching planar G^2 Hermite data. The G^2 Hermite interpolation problem is important because its solution allows one to construct a G^2 Hermite interpolating spline from a set of any number of points, unit tangent vectors, and signed curvatures. To do the construction, one finds the G^2 Hermite interpolating curve between neighbouring pairs of points.

In this paper, a C-shaped interpolating curve made of one or two spirals is sought. Such a curve is considered fair because it comprises a small number of spirals [2, p. 364]. C-shaped curves with zero curvatures in the interior or at the endpoints exist, but are not studied here. Instead, the G^2

* Corresponding author.
E-mail addresses: dereck-meek@umanitoba.ca (D.S. Meek), walton@cs.umanitoba.ca (D.J. Walton).

0377-0427/02/$ - see front matter
PII: S0377-0427(01)00385-5

Hermite data here are restricted to having the curvatures of the same sign. A C-shaped curve made by joining a circular arc and a conic in a G^2 manner is proposed. This curve is a special case of the curve made from a pair of conics in [9]. The pair of conics curve has two free parameters, which can be used as shape parameters. Several difficulties are that the parameters are not directly related to the number of spirals produced, the two conic segments as used there can comprise up to four spiral parts, and the procedure for deciding how many spirals there are in a conic segment is rather complicated [4]. The new C-shaped curve has no free parameters, is unique except for a small region, and it always produces a curve with no more than two spiral parts. Hence, it attempts to produce a fair curve automatically.

Two variations of the new C-shaped curve are the conic/circle curve (a conic followed by a circular arc) and the circle/conic curve (a circular arc followed by a conic). Curves of this type that match given G^2 Hermite data can be found by solving a quadratic equation. The set of planar G^2 Hermite data that can be matched by a single spiral and by a pair of spirals forming a C-shaped curve is known [5,6]. The G^2 Hermite data that this new curve can match is compared to the above theoretical set, to the set that the cubic Bézier curve [1] can match, and to the set that a curve made from a G^2 join of a pair of quadratics [3,9] can match. When a single spiral is required, the cubic Bézier curve appears to cover the widest range of G^2 Hermite data of the three curves. When one or two spirals are used, the new C-shaped curve covers almost the entire range of G^2 Hermite data that is possible and covers a much larger set than either the cubic Bézier curve or the pair-of-quadratics curve.

One of the advantages of the new C-shaped curve is that it guarantees one or two spirals. If one uses a curve such as the cubic Bézier curve for G^2 Hermite interpolation, one can test afterwards to see how many spirals are produced, but it is difficult to force the curve to have only one or two spirals.

2. Preliminaries

2.1. G^2 Hermite data

G^2 Hermite data can be expressed as follows. Let two distinct points be $\mathbf{A}$ and $\mathbf{B}$ where the interpolating curve is to run from $\mathbf{A}$ to $\mathbf{B}$; let the unit tangent vectors at $\mathbf{A}$ and $\mathbf{B}$ be $\mathbf{T}_\mathrm{A}$ and $\mathbf{T}_\mathrm{B}$, where $\mathbf{T}_\mathrm{B}$ is $\mathbf{T}_\mathrm{A}$ rotated by angle W; let the signed curvatures at $\mathbf{A}$ and $\mathbf{B}$ be k_A and k_B. The following restrictions are put on the Hermite data: assume that k_A and k_B are the same sign (zero curvatures are not treated here), and assume that the magnitude of W is less than or equal to $\pi/2$. The sign of W will be the same as the sign of the curvatures. The first restriction is necessary since a C-shaped interpolating curve is required. The second restriction is necessary to ensure that the new curve has only one or two spirals. Segments of parabolas and hyperbolas have at most one curvature extremum, but a segment of an ellipse can have four. The tangents at neighbouring curvature extrema of an ellipse are separated by an angle of $\pi/2$, so restricting the tangent rotation to a magnitude of $\pi/2$ or less means that the conic segment consists of one or two spirals. The cubic Bézier curve can match G^2 Hermite data that has zero curvatures or curvatures of opposite signs, so in one sense it is more general than the curve proposed here. The pair-of-quadratics curve requires that k_A and k_B be nonzero and the same sign.

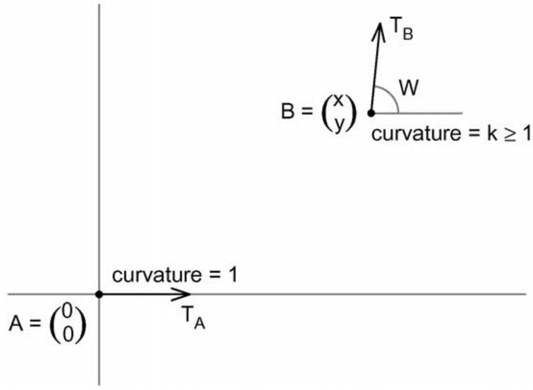

Fig. 1. Standard form for G^2 Hermite data.

A standard form for the G^2 Hermite data can be obtained as follows. Switch **A** and **B** if necessary so that the point with the smaller curvature magnitude is **A**. If a switch is made, the directions of $\mathbf{T}_\mathrm{A}$ and $\mathbf{T}_\mathrm{B}$, the signs of the curvatures, and the sign of W all reverse. If the curvatures are now negative, reflect the data across the X-axis so that the curvatures and W become positive. Translate **A** to the origin and scale the plane by the factor k_A. This scaling causes the curvature at **A** to become 1 and the curvature at **B** to become $k = k_\mathrm{B}/k_\mathrm{A}$, which is a number greater than or equal to 1. Rotate the plane so that $\mathbf{T}_\mathrm{A}$ falls on the positive X-axis (see Fig. 1). **B** will be in the first quadrant between the X-axis and the line $(\cos W)y = (\sin W)x$. When **B** is considered as a given fixed point (Sections 2 and 3), it will be represented by $(x_\mathrm{B}, y_\mathrm{B})^\mathrm{T}$, but when it is considered a variable point (Section 4), it will be represented by $(x, y)^\mathrm{T}$. The G^2 Hermite data are now expressed in terms of the four variables x_B, y_B, W, and k.

2.2. Reachable regions

Since a C-shaped curve is required for interpolation, only a restricted set of the G^2 Hermite data can be matched. One way to show the G^2 Hermite data that can be matched by a C-shaped curve is to put the Hermite data in the above standard form and consider different positions for **B**. For some positions, there may be one or more C-shaped curves while for others, there may be none. For example when **B** is below the X-axis, there is no C-shaped curve matching the G^2 Hermite data in standard form. The region made of the positions of **B** for which one or more C-shaped curves exist will be referred to as the region that can be reached by a C-shaped curve. This representation has the advantage of showing what is possible as a geometric region (rather than as conditions on the curvature values or unit tangent vectors, which are harder to visualize).

The region that can be reached by a single spiral whose curvature increases from 1 to k and whose tangent vector rotates through an angle W appears in [6]. This region is shown in Fig. 2. The definitions of points, lines, and curves that appear in this and other figures are given in the appendix. If $k = 1$, this region reduces to the single point **D** (points **D** and **E** coincide). The single

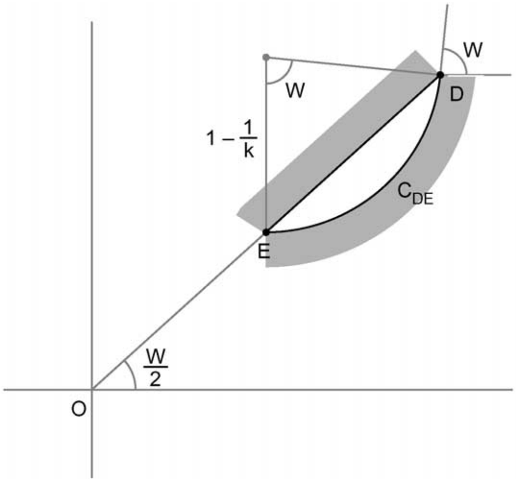

Fig. 2. Reachable region for a single spiral of increasing positive curvature.

spiral is then the unique circular arc that matches **A**, $\mathbf{T}_A$, with radius 1 whose unit tangent vector rotates through W.

The region that can be reached with a C-shaped curve made of a pair of spirals whose curvature is 1 at the beginning of the first spiral and k at the end of the second spiral and whose tangent vector rotates through an angle W was analysed in [5]. The region for two-spiral curves includes the region that can be reached with a single spiral and it will be referred to as the region reached by one or two spirals (see Fig. 3). When $k = 1$, the region is particularly simple (again, points **D** and **E** coincide).

2.3. Conic

A segment of a conic can be expressed as a quadratic rational function of the form

$$B(t) = \frac{(1-t)^2\mathbf{B}_0 + 2w(1-t)t\mathbf{B}_1 + t^2\mathbf{B}_2}{(1-t)^2 + 2w(1-t)t + t^2}, \quad 0 \leqslant t \leqslant 1, \ 0 < w < \infty,$$

where the $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are three control points and w is a weight parameter that determines which conic curve is represented [2, p. 199] (see Fig. 4). Let θ be the angle from $\mathbf{B}_1 - \mathbf{B}_0$ to $\mathbf{B}_2 - \mathbf{B}_1$. To avoid degenerate cases, assume that $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are distinct and that θ is in $(0, \pi)$. Forcing the curvature to be k_A at $t = 0$ and k_B at $t = 1$ gives the two equations

$$k_A = \frac{(\mathbf{B}_1 - \mathbf{B}_0) \times (\mathbf{B}_2 - \mathbf{B}_1)}{2w^2\|\mathbf{B}_1 - \mathbf{B}_0\|^3} \quad \text{and} \quad k_B = \frac{(\mathbf{B}_1 - \mathbf{B}_0) \times (\mathbf{B}_2 - \mathbf{B}_1)}{2w^2\|\mathbf{B}_2 - \mathbf{B}_1\|^3},$$

where $\times$ stands for the two-dimensional cross product, $(x_0, y_0)^T \times (x_1, y_1)^T = x_0y_1 - x_1y_0$. The quotient of these two equations shows that the lengths of the segments in the control polyline are in a

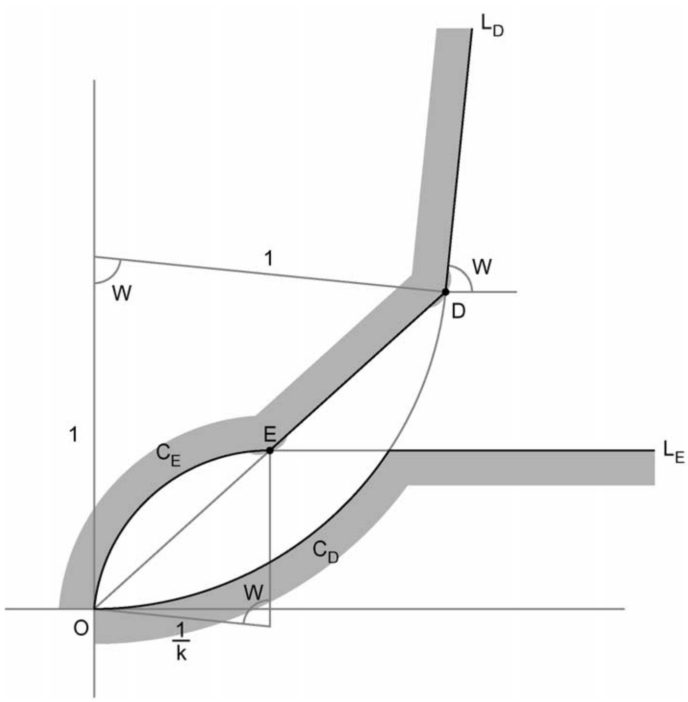

Fig. 3. Reachable region for one or two spirals of positive curvature forming a C-shaped curve.

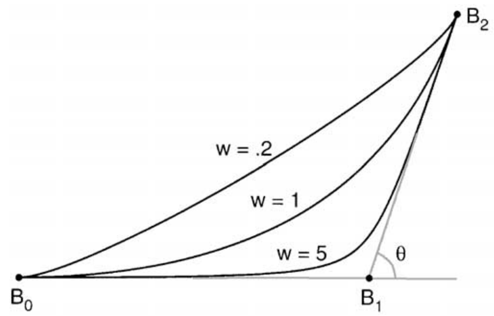

Fig. 4. Rational quadratic (conic) with various weight parameters.

proportion determined by the curvatures at the endpoints,

$$\frac{\|\mathbf{B}_2-\mathbf{B}_1\|}{\|\mathbf{B}_1-\mathbf{B}_0\|}=\left(\frac{k_{\mathrm{A}}}{k_{\mathrm{B}}}\right)^{1/3}. \tag{2.1}$$

The weight w can be calculated from the three control points and the curvatures with

$$w=\sqrt{\frac{(\mathbf{B}_1-\mathbf{B}_0)\times(\mathbf{B}_2-\mathbf{B}_1)}{2k_{\mathrm{A}}\|\mathbf{B}_1-\mathbf{B}_0\|^3}}=\sqrt{\frac{\sin\theta}{2k_{\mathrm{A}}^{2/3}k_{\mathrm{B}}^{1/3}\|\mathbf{B}_1-\mathbf{B}_0\|}}. \tag{2.2}$$

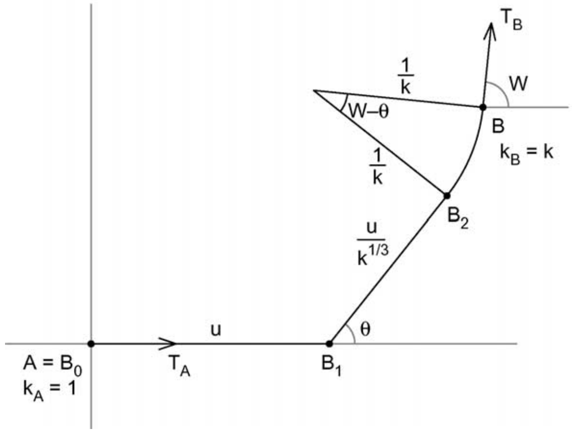

Fig. 5. Conic/circle curve.

3. Conic/circle and circle/conic curves

3.1. Finding the conic/circle curve that matches G^2 Hermite data

With the G^2 Hermite data in standard form, $\mathbf{A}$ is the origin and $\mathbf{B}$ is in the first quadrant. Let the control points of the conic part of the curve be $\mathbf{B}_0 = \mathbf{A}$, $\mathbf{B}_1$ on the X-axis, and $\mathbf{B}_2$. By (2.1), curvatures 1 and k at $\mathbf{B}_0$ and $\mathbf{B}_2$ determine the ratio of the segments in the control polyline. Control points $\mathbf{B}_1$ and $\mathbf{B}_2$ can be expressed in terms of the two unknowns u and θ where $u = \|\mathbf{B}_1 - \mathbf{B}_0\|$ and θ is the angle from $\mathbf{B}_1 - \mathbf{B}_0$ to $\mathbf{B}_2 - \mathbf{B}_1$ (see Fig. 5). For a sensible solution, u must be positive and θ must be in $(0, W)$. One can allow θ to be W, in which case the curve has just the conic part. This special case will be referred to later but not treated in detail. With these unknowns,

$$\mathbf{B}_1 = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \begin{pmatrix} u \\ 0 \end{pmatrix} + \frac{u}{k^{1/3}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Adding the circular arc to $\mathbf{B}_2$ and the fact that $\mathbf{B}$ is $(x_{\mathrm{B}}, y_{\mathrm{B}})^{\mathrm{T}}$ gives two equations for u and θ:

$$\mathbf{B} = \begin{pmatrix} u \\ 0 \end{pmatrix} + \frac{u}{k^{1/3}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \frac{1}{k} \begin{pmatrix} -\sin\theta + \sin W \\ \cos\theta - \cos W \end{pmatrix} = \begin{pmatrix} x_{\mathrm{B}} \\ y_{\mathrm{B}} \end{pmatrix},$$

or

$$\left(1 + \frac{\cos\theta}{k^{1/3}}\right) u + \frac{1}{k}(-\sin\theta + \sin W) = x_{\mathrm{B}} \tag{3.1}$$

and

$$\left(\frac{\sin\theta}{k^{1/3}}\right) u + \frac{1}{k}(\cos\theta - \cos W) = y_{\mathrm{B}}. \tag{3.2}$$

Eliminate u to give a single equation in θ,

$$(-kx_{\mathrm{B}} + \sin W)\sin\theta + (ky_{\mathrm{B}} - k^{1/3} + \cos W)\cos\theta + k^{4/3}y_{\mathrm{B}} + k^{1/3}\cos W - 1 = 0. \tag{3.3}$$

Replacement of θ by $t = \tan\theta/2$ transforms (3.3) into the quadratic equation

$$q(t) = (k^{1/3} - 1)(ky_{\mathrm{B}} + 1 + \cos W)t^2 + 2(-kx_{\mathrm{B}} + \sin W)t$$
$$+(k^{1/3} + 1)(ky_{\mathrm{B}} - 1 + \cos W) = 0. \tag{3.4}$$

The interval for t which corresponds to θ in $(0, W)$ is

$$T = (0, \tan\tfrac{W}{2}) = \left(0, \frac{1 - \cos W}{\sin W}\right). \tag{3.5}$$

Note that for $\mathbf{B}$ in the first quadrant,

$$ky_{\mathrm{B}} + 1 + \cos W > 0. \tag{3.6}$$

If $k = 1$, $y = q(t)$ is a line in the t–y plane; if $k > 1$, (3.6) shows that $y = q(t)$ is an up-opening parabola. These observations are used extensively in Section 4 in the determination of the number of roots of (3.4). The endpoint values of $q(t)$ are

$$q(0) = (k^{1/3} + 1)(ky_{\mathrm{B}} - 1 + \cos W) \tag{3.7}$$

and

$$q(\tan(W/2)) = \frac{2k}{(1 + \cos W)}[-(\sin W)x_{\mathrm{B}} + (k^{1/3} + \cos W)y_{\mathrm{B}}]. \tag{3.8}$$

The procedure for finding a conic/circle curve that matches G^2 Hermite data is as follows. With a given $\mathbf{B}$, W, and k that express G^2 Hermite data in standard form, find the roots of (3.4). For each real root in T, a corresponding value for θ can be determined. Either (3.1) or (3.2) gives a value for u. From (3.2) on replacing θ by $t = \tan(\theta/2)$ and using (3.6), the requirement that u is positive becomes

$$t^2 > \frac{-ky_{\mathrm{B}} + 1 - \cos W}{ky_{\mathrm{B}} + 1 + \cos W}. \tag{3.9}$$

In short, any solution to (3.4) in interval T of (3.5) which satisfies (3.9) is acceptable. The corresponding θ and u can be used to find the three control points of the quadratic rational. The weight parameter can be found from (2.2), with $k_{\mathrm{A}} = 1$, $k_{\mathrm{B}} = k$. The centre of the circular arc can be found from point $\mathbf{B}$, unit tangent vector $\mathbf{T}_{\mathrm{B}}$, and its radius is $1/k$; the angle of the arc is $W - \theta$.

3.2. Finding the circle/conic curve that matches G^2 Hermite data

The circle/conic curve can be obtained from the conic/circle analysis by asking for a curve whose ending curvature is less than or equal to its beginning curvature, $0 < k \leqslant 1$. However, in examination of the reachable regions, it is perhaps less confusing to develop the two curves separately. A brief summary of the calculations required to find the circle/conic curve follows (see Fig. 6). Corresponding

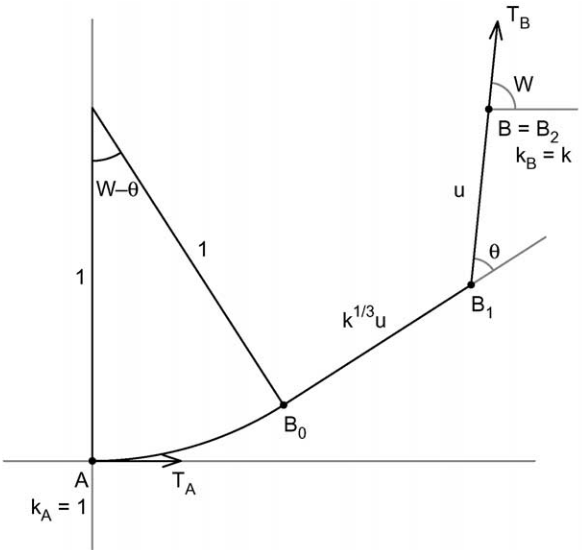

Fig. 6. Circle/conic curve.

to Eqs. (3.1) and (3.2) are

$$[k^{1/3}\cos(W-\theta)+\cos W]u+\sin(W-\theta)=x_{\rm B} \tag{3.1*}$$

and

$$[k^{1/3}\sin(W-\theta)+\sin W]u+1-\cos(W-\theta)=y_{\rm B}. \tag{3.2*}$$

Eliminate u to give a single equation in θ,

$$\begin{aligned}&-k^{1/3}[(\cos W)x_{\rm B}+(\sin W)(y_{\rm B}-1)\sin\theta\\ &+[k^{1/3}(\sin W)x_{\rm B}-k^{1/3}(\cos W)(y_{\rm B}-1)-1]\cos\theta\\ &+(\sin W)x_{\rm B}-(\cos W)(y_{\rm B}-1)-k^{1/3}=0.\end{aligned} \tag{3.3*}$$

Corresponding to Eq. (3.4) is

$$\begin{aligned}q^*(t)=&-(k^{1/3}-1)[(\sin W)x_{\rm B}-(\cos W)(y_{\rm B}-1)+1]t^2\\ &-2k^{1/3}[(\cos W)x_{\rm B}+(\sin W)(y_{\rm B}-1)]t\\ &+(k^{1/3}+1)[(\sin W)x_{\rm B}-(\cos W)(y_{\rm B}-1)-1].\end{aligned} \tag{3.4*}$$

B is between the X-axis and the line $(\cos W)y=(\sin W)x$, so

$$(\sin W)x_{\rm B}-(\cos W)(y_{\rm B}-1)+1>0. \tag{3.6*}$$

If $k=1$, $y=q^*(t)$ is a line; if $k>1$, (3.6*) shows that $y=q^*(t)$ is a down-opening parabola. These observations are used extensively in Section 4 in the determination of the number of roots of (3.4*).

The endpoint values of $q^*(t)$ are

$$q^*(0) = (k^{1/3} + 1)[(\sin W)x_B - (\cos W)(y_B - 1) - 1] \tag{3.7*}$$

and

$$q^*(\tan(W/2)) = \frac{2}{1+\cos W}[(\sin W)x_B - (k^{1/3} + \cos W)y_B]. \tag{3.8*}$$

A solution to (3.4*) in interval T of (3.5) is acceptable if it makes u positive. The combination $\sin W$ times (3.1*) minus $\cos W$ times (3.2*) gives

$$k^{1/3}(\sin\theta)u + \cos\theta = (\sin W)x_B - (\cos W)(y_B - 1).$$

Replacing θ by $t = \tan(\theta/2)$ in the above and using (3.6*), the condition for u positive becomes

$$t^2 > \frac{-(\sin W)x_B + (\cos W)(y_B - 1) + 1}{(\sin W)x_B - (\cos W)(y_B - 1) + 1}. \tag{3.9*}$$

If an acceptable t has been found, calculate θ and u. The control points $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}_2$ can now be found; the weight parameter comes from (2.1). The centre of the circular arc is $(0,1)^T$, its radius is 1, and its angle is $W - \theta$.

4. Reachable region of conic/circle and circle/conic curves

The region that can be reached by a G^2 Hermite interpolating curve made of one or two spirals forming a C-shaped curve is shown in Fig. 3 (see Section 2 for further details). Remarkably, almost all of that region can be reached by a unique conic/circle or a unique circle/conic curve. There is a small region which can be reached by two conic/circle curves and a small region which can be reached by two circle/conic curves. There is also a small region where C-shaped curves of one or two spirals are possible, but the conic/circle and the circle/conic curves cannot reach it. It is possible to do the analysis of the conic/circle and circle/conic curves together because they both consist of a conic joined to a circle. However, the discussion seems more clear if they are done separately. The analysis reduces to finding the $\mathbf{B}$ for which there are acceptable solutions to (3.4) or (3.4*). In this section $\mathbf{B}$ is considered a variable, so it will be represented by $(x, y)^T$ rather than the previously used $(x_B, y_B)^T$.

4.1. Conic/circle curve

Consider the four cases that arise from having the signs of $q(t)$ at the endpoints of T take the four possible combinations.

Case 1: Region for $\mathbf{B}$ when $q(0) > 0$ and $q(\tan(W/2)) < 0$, Fig. 7.

The change of sign of $q(t)$ means Eq. (3.4) has one root in T of (3.5). From (3.7), $q(0) > 0$ implies

$$ky > 1 - \cos W \quad (\mathbf{B} \text{ above line } L_E), \tag{4.1}$$

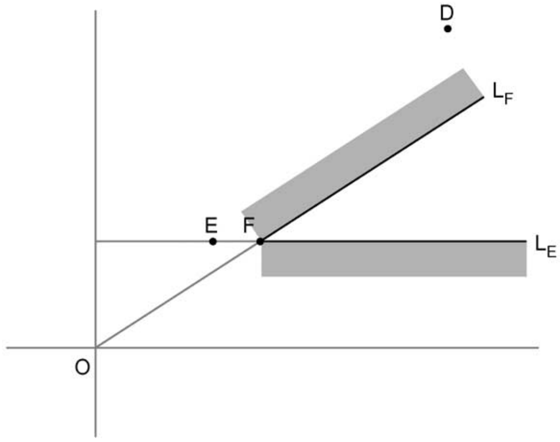

Fig. 7. Case 1: unique conic/circle curve when $q(0) > 0$ and $q(\tan \frac{W}{2}) < 0$.

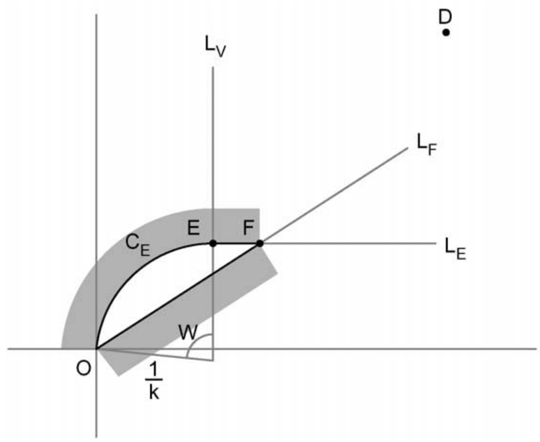

Fig. 8. Case 2: unique conic/circle curve when $q(0) < 0$ and $q(\tan \frac{W}{2}) > 0$.

and (4.1) implies that the numerator of the right-hand side of (3.9) is negative, so (3.9) is satisfied. From (3.8), $q(\tan(w/2)) < 0$ implies that

$$y < \left(\frac{\sin W}{k^{1/3} + \cos W}\right) x \quad (\mathbf{B} \text{ below line } L_{\mathrm{F}}). \tag{4.2}$$

In conclusion, there is a unique G^2 Hermite interpolating conic/circle curve when **B** is in the region between lines L_{E} and L_{F} as shown in Fig. 7.

Case 2: Region for **B** when $q(0) < 0$ and $q(\tan(W/2)) > 0$, Fig. 8.

The change in sign of $q(t)$ means Eq. (3.4) has one root in T of (3.5). From (3.7), $q(0) < 0$ implies

$$ky < 1 - \cos W \quad (\mathbf{B} \text{ below line } L_{\mathrm{E}}) \tag{4.3}$$

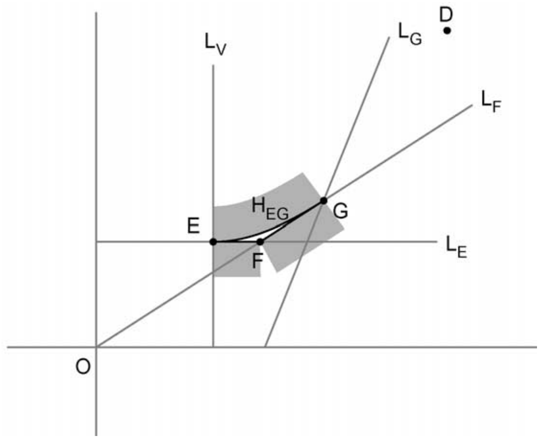

Fig. 9. Case 3: two conic/circle curves when $q(0) > 0$, $q(\tan\frac{W}{2}) > 0$, (4.8), (4.9), and (4.10).

and (4.3) implies that (3.9) gives a positive lower limit on t of

$$t_{\mathrm{L}} = \sqrt{\frac{-ky + 1 - \cos W}{ky + 1 + \cos W}}. \tag{4.4}$$

$q(t_{\mathrm{L}}) < 0$ means both that $t_{\mathrm{L}} < \tan(W/2)$ and there is one root of (3.4) in $(t_{\mathrm{L}}, \tan(W/2))$, while $q(t_{\mathrm{L}}) \geqslant 0$ means there are no acceptable roots of (3.4). Calculation shows

$$q(t_{\mathrm{L}}) = 2[ky - 1 + \cos W + (-kx + \sin W)t_{\mathrm{L}}]. \tag{4.5}$$

If $kx \geqslant \sin W$ (**B** on or right of line L_{V}), then (4.3) and (4.5) show $q(t_{\mathrm{L}}) < 0$. If $kx < \sin W$ (**B** left of line L_{V}), then

$$(kx - \sin W)^2 + (ky + \cos W)^2 < 1 \quad (\mathbf{B} \text{ inside circle } C_{\mathrm{E}})$$

is required to make $q(t_{\mathrm{L}}) < 0$. Regardless of the position of **B** with respect to L_{V}, from (3.8) $q(\tan(W/2)) > 0$ implies that

$$y > \left(\frac{\sin W}{k^{1/3} + \cos W}\right) x \quad (\mathbf{B} \text{ above line } L_{\mathrm{F}}). \tag{4.6}$$

In conclusion, there is a unique G^2 Hermite interpolating conic/circle curve when **B** is in the region shown in Fig. 8.

Case 3: Region for **B** when $q(0) > 0$ and $q(\tan(W/2)) > 0$, Fig. 9.

If $k = 1$, Eq. (3.4) has no roots in T of (3.5).

If $k > 1$, Eq. (3.4) has zero or two roots in T. As in Case 1, $q(0) > 0$ implies (4.1), so (3.9) is satisfied. As in Case 2, $q(\tan(W/2)) > 0$ implies (4.6). The t-value for which the minimum of the parabola $y = q(t)$ occurs is

$$t_{\mathrm{M}} = \frac{kx - \sin W}{(k^{1/3} - 1)(ky + 1 + \cos W)}. \tag{4.7}$$

There will be two roots of (3.4) in T if and only if t_{M} is in T and $q(t_{\mathrm{M}}) \leqslant 0$. t_{M} in T implies

$$kx > \sin W \quad (\mathbf{B} \text{ right of line } L_{\mathrm{V}}) \tag{4.8}$$

and

$$(k^{1/3} - 1)y > \frac{\sin W}{1 - \cos W}x - \frac{1 + \cos W}{k^{2/3}} \quad (\mathbf{B} \text{ above line } L_{\mathrm{G}}). \tag{4.9}$$

Calculation shows

$$q(t_{\mathrm{M}}) = \frac{k^{1/3} + 1}{ky + 1 + \cos W}\left[-\frac{(kx - \sin W)^2}{k^{2/3} - 1} + (ky + \cos W)^2 - 1\right],$$

from which $q(t_{\mathrm{M}}) \leqslant 0$ implies

$$(ky + \cos W)^2 \leqslant \frac{(kx - \sin W)^2}{k^{2/3} - 1} + 1 \tag{4.10}$$

(**B** on or below the upper branch of hyperbola H_{EG}). In conclusion, there are two G^2 Hermite interpolating conic/circle curves when **B** is in the region shown in Fig. 9. The two curves coincide when **B** is between **E** and **G** on hyperbola H_{EG}.

The inequality

$$\frac{1 - \cos W}{k \sin W} \leqslant \frac{\sin W}{k^{2/3}(1 + k^{1/3}\cos W)}$$

shows that points **F** and **G** are in the order shown in Fig. 9 on line L_{F} (**F** and **G** coincide if and only if $k = 1$). The hyperbola H_{EG} is tangent to L_{E} at **E** and tangent to L_{F} at **G**.

Case 4: Region for **B** when $q(0) < 0$ and $q(\tan(W/2)) < 0$.

If $k = 1$, Eq. (3.4) has no roots in T of (3.5); if $k > 1$, Eq. (3.4) also has no roots in T. There is no region for **B** in this case.

4.2. Circle/conic curve

The analysis for the circle/conic is similar to the preceding analysis for the conic/circle. The details are contained in [7] and only results are reported here. As in the conic/circle analysis, there are four cases that arise from having the signs of $q^*(t)$ at the endpoints of T take the four possible combinations.

Case 1*: Region for **B** when $q^*(0) > 0$ and $q^*(\tan(W/2)) < 0$, Fig. 10.

There is a unique G^2 Hermite interpolating circle/conic curve when **B** is between lines L_{F} and L_{D} as shown in Fig. 10.

Case 2*: Region for **B** when $q^*(0) < 0$ and $q^*(\tan(W/2)) > 0$, Fig. 11.

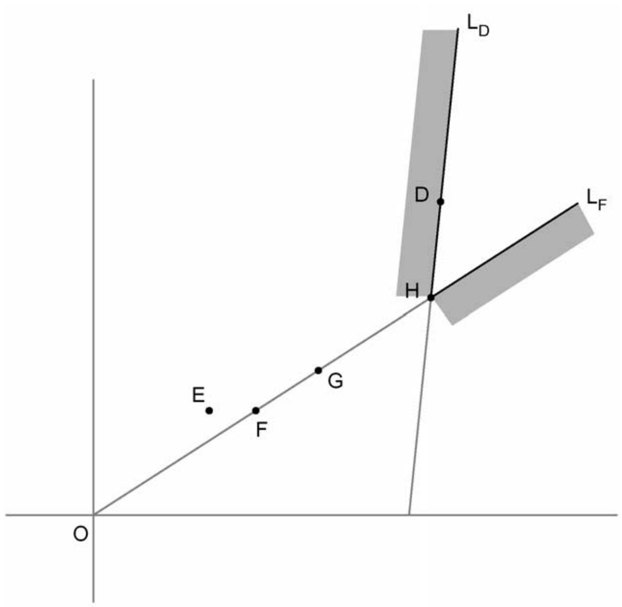

Fig. 10. Case 1*: unique circle/conic curve when $q^*(0) > 0$ and $q^*(\tan(W/2)) < 0$.

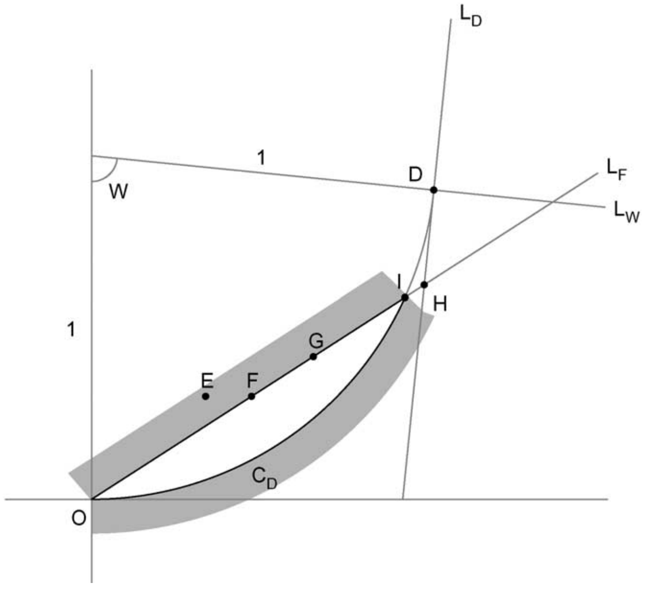

Fig. 11. Case 2*: unique circle/conic curve when $q^*(0) < 0$ and $q^*(\tan(W/2)) > 0$.

There is a unique G^2 Hermite interpolating circle/conic curve when **B** is in the region bounded by an arc of circle C_D and line segment **O**I shown in Fig. 11.

Case 3*: Region for **B** when $q^*(0) > 0$ and $q^*(\tan(W/2)) > 0$.

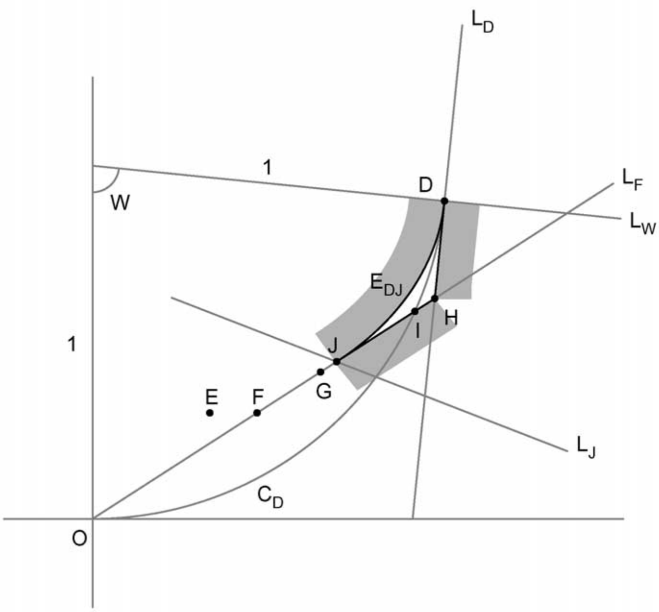

Fig. 12. Case 4*: circle/conic curves when $q^*(0) < 0$, $q^*(\tan(W/2)) < 0$; unique curve when **B** is in region **DIH**, two curves when **B** is in region **DJI**.

There is no region for **B** in this case.

Case 4*: Region for **B** when $q^*(0) < 0$ and $q^*(\tan(W/2)) < 0$.

There is a unique G^2 Hermite interpolating circle/conic curve when **B** is in region **DIH**, and there are two G^2 Hermite interpolating circle/conic curves when **B** is in region **DJI** as shown in Fig. 12.

4.3. Both conic/circle and circle/conic curves

The union of the regions in Figs. 7–12 plus the line segments EF and DH, and line L_F in the first quadrant give the region for **B** for which a conic/circle or a circle/conic curve can provide a G^2 Hermite interpolating curve that is C-shaped and consists of one or two spirals (see Fig. 13). If **B** is on line segment EF, (3.4) has one acceptable nonzero root so there is a G^2 Hermite interpolating conic/circle curve. If **B** is on line segment DH, (3.4*) has one acceptable nonzero root so there is a G^2 Hermite interpolating circle/conic curve. Line L_F in the first quadrant represents the special case $\theta = W$ where the G^2 Hermite interpolating curve is a conic without an adjoining circular arc.

5. Comparison of reachable regions of various curves

The reachable regions for a G^2 Hermite interpolating cubic Bézier curve and for a pair-of-quadratics curve would be difficult to find analytically. In this section these regions and the region for the new C-shaped curve will be illustrated numerically. To create an illustration of a reachable region, choose a large range of **B** points in the first quadrant, find the appropriate curve, and test whether it consists

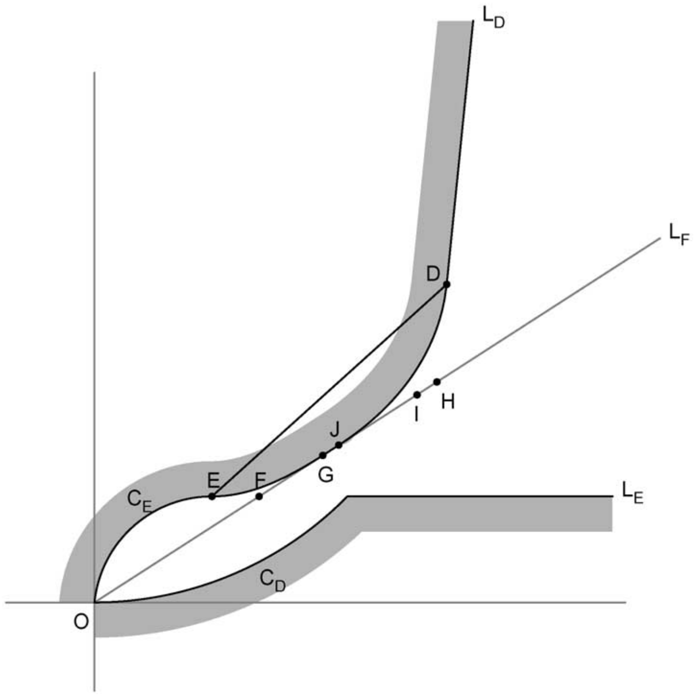

Fig. 13. The reachable region for conic/circle and circle/conic curves.

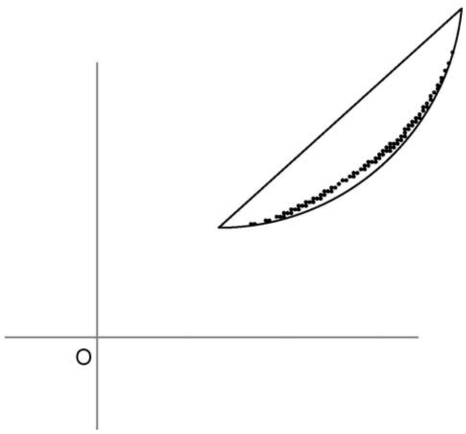

Fig. 14. The reachable region for conic/circle and circle/conic curve of one spiral.

of one or two spirals. If it does, place a heavy dot at the position of **B**; the collection of dots will indicate a region. One cannot be completely confident of such numerical results, but they probably give a good indication of the reachable regions. The regions for C-shaped curves of one spiral and the theoretical region of Fig. 2 are shown in Figs. 14 and 15; the regions for C-shaped curves of one or two spirals and the theoretical region of Fig. 3 are shown in Figs. 16–18.

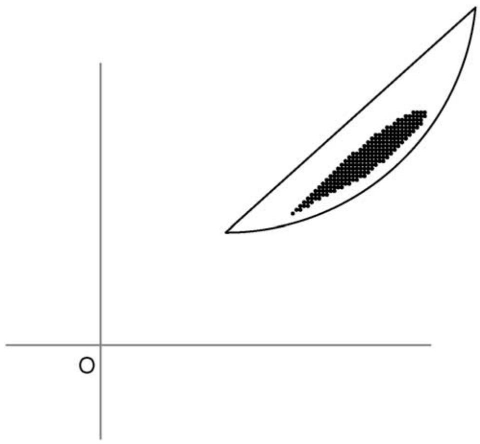

Fig. 15. The reachable region for a cubic Bézier curve of one spiral.

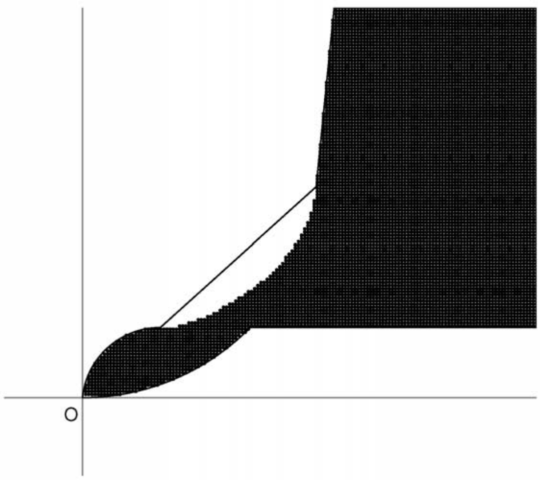

Fig. 16. The reachable region for conic/circle and circle/conic curves of one or two spirals.

5.1. C-shaped curve of one spiral

A selection of points (indicated by heavy dots) that can be reached by a single spiral conic/circle or a single spiral circle/conic curve is shown in Fig. 14. A fairly lengthy calculation for deciding whether a conic segment is one spiral is given in [4], so it may be possible to determine this region analytically. However, the numerical results suggest that the reachable region is not a large portion of the theoretical region.

The reachable region for the G^2 Hermite interpolating cubic Bézier curve which is a single spiral is indicated in Fig. 15. This region is considerably larger than the region reached by the new curve of one spiral, but it is still not a very large part of the theoretical region.

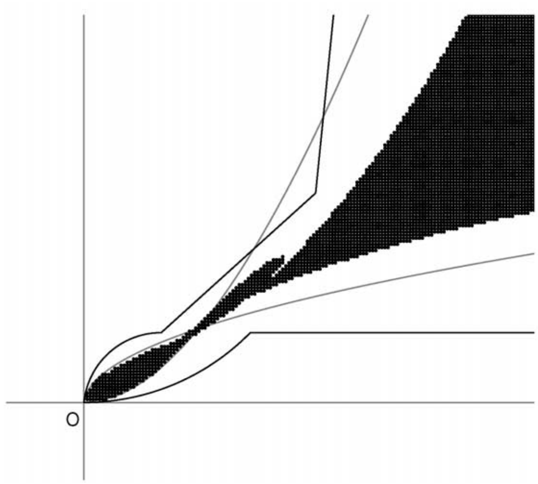

Fig. 17. The reachable region for a cubic Bézier curve of one or two spirals.

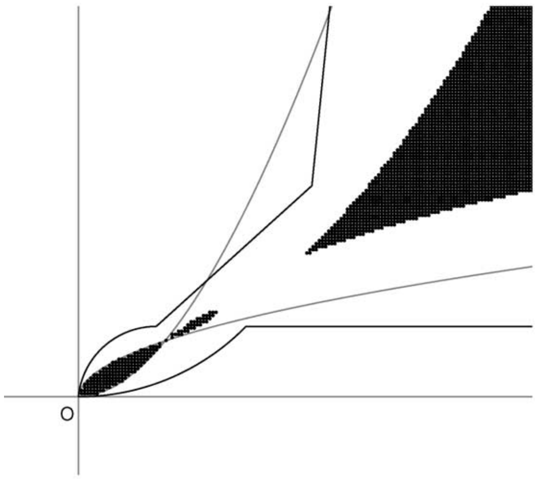

Fig. 18. The reachable region for a pair-of-quadratics curve of two spirals.

Numerical tests never produced a G^2 Hermite interpolating pair-of-quadratics curve that was a single spiral. Consequently, no figure is shown.

5.2. C-shaped curve of one or two spirals

The reachable region of a G^2 Hermite interpolating C-shaped conic/circle and circle/conic curves of one or two spirals was found analytically in Section 4 and shown in Fig. 13. The G^2 Hermite interpolating conic/circle and circle/conic curves of one or two spirals cover almost the entire region that can be reached by a C-shaped curve of one or two spirals (see Fig. 3). The only missing part is the D-shaped region **DEGJ** shown in Fig. 13. A numerical demonstration of this reachable region is shown in Fig. 16.

The reachable region for the G^2 Hermite interpolating cubic Bézier curve which consists of one or two spirals is indicated in Fig. 17. Two simple conditions in the notation of [1] for a unique G^2 Hermite interpolating cubic Bézier curve are (i) $0 < R_0 < 1$ and $0 < R_1 < 1$ or (ii) $1 < R_0$ and $1 < R_1$. In the notation used here

$$R_0 = \frac{3}{2}\frac{[\sin W)x - (\cos W)y]^2}{(\sin^2 W)y}, \quad R_1 = \frac{3}{2}\frac{ky^2}{(\sin^2 W)[(\sin W)x - (\cos W)y]}.$$

Conditions (i) and (ii) describe the regions between the parabolas that are drawn lightly in Fig. 17. Of course the unique solution is not always a curve of one or two spirals.

The reachable region for the G^2 Hermite interpolating pair-of-quadratics curve which consists of two spirals is indicated in Fig. 18. Using the notation in [3], there is a unique G^2 Hermite interpolating pair-of-quadratics curve if and only if (iii) $0 < \lambda_0 < 1$ and $0 < \lambda_1 < 1$ or (iv) $1 < \lambda_0$ and $1 < \lambda_1$. In the notation used here

$$\lambda_0 = 2\frac{[(\sin W)x - (\cos W)y]^2}{(\sin^2 W)y}, \quad \lambda_1 = 2\frac{ky^2}{(\sin^2 W)[(\sin W)x - (\cos W)y]}.$$

Conditions (iii) and (iv) describe the regions between the parabolas that are drawn lightly in Fig. 18. Of course the unique solution is not always a curve of one or two spirals.

6. Algorithm

A series of G^2 Hermite data could be denoted $\mathbf{A}$, $\mathbf{T}_\mathrm{A}$, k_A, $\mathbf{B}$, $\mathbf{T}_\mathrm{B}$, $k_\mathrm{B}, \mathbf{C}$, $\mathbf{T}_\mathrm{C}$, $k_\mathrm{C}, \ldots$. For each neighbouring pair, one can attempt to find a G^2 Hermite interpolating curve segment. The collection of all such segments will be a piecewise curve that has G^2 smoothness and that interpolates the given G^2 Hermite data. For the discussion below, consider just the first G^2 Hermite interpolating segment.

If $k_\mathrm{A}k_\mathrm{B} < 0$, a C-shaped curve cannot fit the data; an S-shaped curve might be appropriate.

If $k_\mathrm{A}\ k_\mathrm{B} = 0$, a C-shaped curve may exist, but will not be found by the method in this paper. This omission is necessary since conics are used here and they cannot have points with zero curvature.

If the angle W from T_A to T_B has magnitude greater than $\pi/2$, the method here cannot guarantee at most two spirals.

With the assumptions that $k_\mathrm{A}\ k_\mathrm{B} > 0$ and $|W| < \pi/2$, the two-point G^2 Hermite data can be transformed into the standard form in Section 2. If the transformed $\mathbf{B}$ is in the reachable region in Fig. 3, then a C-shaped curve of one or two spirals is possible, otherwise it is not. If $\mathbf{B}$ is in the region in Fig. 3 and also in the lens-shaped region **DEGJ** in Fig. 13, then there is no conic/circle or circle/conic curve that can match the G^2 Hermite data.

Assuming $\mathbf{B}$ is in the reachable region indicated in Fig. 13, if $\mathbf{B}$ is in the small regions **EFG** or **DJI**, there are two solutions. One could perhaps choose the solution with the smaller angle in the circular arc part. If $\mathbf{B}$ is elsewhere in the region in Fig. 13, there is a unique solution. If $\mathbf{B}$ is in the union of the regions in Figs. 7–9, use the formulas for a conic/circle (first part of Section 3); if $\mathbf{B}$ is in the union of the regions in Figs. 10–12, use the formulas for a circle/conic (second part of Section 3).

In the above cases where a C-shaped curve of one or two spirals of the type described here does not exist, a user may change the Hermite data, add or remove Hermite data, or try to find another type of curve such as an S-shaped curve [8].

7. Conclusions

The conic/circle and circle/conic curves proposed here are C-shaped curves of one or two spirals that can match almost any G^2 Hermite data that can be matched by a C-shaped curve of one or two spirals. The G^2 Hermite interpolating conic/circle and circle/conic curves can be found by solving a quadratic equation. Although a unique solution is usual, two acceptable solutions are sometimes found.

From Figs. 14 and 15, it appears the cubic Bézier curve with a single spiral has a larger reachable region than the conic/circle and circle/conic curves with one spiral. From Figs. 16 and 17, it appears the cubic Bézier curve with one or two spirals has a much smaller reachable region than the conic/circle and circle/conic curves with one or two spirals. The G^2 Hermite interpolating cubic Bézier curve can be found by solving a pair of quadratic equations or a single quartic equation. Although a unique solution is usual, two or three solutions are sometimes found. One difficulty is that one can only tell afterwards whether a cubic Bézier curve that matches given G^2 Hermite data comprises one or two spirals.

From Figs. 17 and 18, it appears the pair-of-quadratics curve with two spirals has a reachable region similar to that of the cubic Bézier curve of one or two spirals, and much smaller than that of the new C-shaped curves. The G^2 Hermite interpolating pair-of-quadratics curve can be found by solving a quadratic equation. The solution is always unique. As with cubic Bézier curves, one can only tell afterwards whether a pair-of-quadratics curve that matches given G^2 Hermite data comprises one or two spirals.

Acknowledgements

The authors acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada for this research.

Appendix

Points:

$$\mathbf{D} = \begin{pmatrix} \sin W \\ 1 - \cos W \end{pmatrix},$$

$$\mathbf{E} = \frac{1}{k} \begin{pmatrix} \sin W \\ 1 - \cos W \end{pmatrix},$$

$$\mathbf{F} = \frac{1 - \cos W}{k \sin W} \begin{pmatrix} k^{1/3} + \cos W \\ \sin W \end{pmatrix},$$

$$\mathbf{G} = \frac{\sin W}{k^{2/3}(1 + k^{1/3}\cos W)} \begin{pmatrix} k^{1/3} + \cos W \\ \sin W \end{pmatrix},$$

$$\mathbf{H} = \frac{1 - \cos W}{k^{1/3}\sin W} \begin{pmatrix} k^{1/3} + \cos W \\ \sin W \end{pmatrix},$$

$$\mathbf{I} = \frac{2\sin W}{k^{2/3} + 2k^{1/3}\cos W + 1} \begin{pmatrix} k^{1/3} + \cos W \\ \sin W \end{pmatrix},$$

$$\mathbf{J} = \frac{\sin W}{k^{1/3}(k^{1/3} + \cos W)} \begin{pmatrix} k^{1/3} + \cos W \\ \sin W \end{pmatrix}.$$

Lines:

L_{D}: $(\cos W)y = (\sin W)x - 1 + \cos W$ (vertical line if $W = \pi/2$),

L_{E}: $ky = 1 - \cos W$ (horizontal line),

L_{F}: $y = \left(\dfrac{\sin W}{k^{1/3} + \cos W}\right) x$,

L_{G}: $(k^{1/3} - 1)\ y = \dfrac{\sin W}{1 - \cos W}x - \dfrac{1 + \cos W}{k^{2/3}}$,

L_{J}: $(k^{1/3} + \cos W)\ y = -\dfrac{\sin W}{1 - \cos W}(k^{1/3} - 1 + \cos W)x + 1 + \cos W$,

L_{V}: $kx = \sin W$ (vertical line),

L_{W}: $(\sin W)y = -(\cos W)x + \sin W$ (horizontal line if $W = \pi/2$).

Circles:

C_{D}: $x^2 + (y - 1)^2 = 1$, centre : $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, radius : 1,

C_{E}: $(kx - \sin W)^2 + (ky + \cos W)^2 = 1$, centre : $\dfrac{1}{k}\begin{pmatrix} \sin W \\ -\cos W \end{pmatrix}$, radius : $\dfrac{1}{k}$,

C_{DE}: $(kx - \sin W)^2 + (ky - k + \cos W)^2 = (k - 1)^2$,

centre : $\dfrac{1}{k}\begin{pmatrix} \sin W \\ k - \cos W \end{pmatrix}$, radius : $1 - \dfrac{1}{k}$.

Ellipse:

$$E_{\mathrm{DJ}}\colon\ \frac{k^{2/3}}{k^{2/3}-1}[(\cos W)x+(\sin W)(y-1)]^2+[(\sin W)x-(\cos W)(y-1)]^2=1,$$

$$\text{centre:}\begin{pmatrix}0\\1\end{pmatrix},\quad \text{major axis : } L_{\mathrm{W}}.$$

Hyperbola:

$$H_{\mathrm{EG}}\colon\quad (ky+\cos W)^2=\frac{(kx-\sin W)^2}{k^{2/3}-1}+1,\quad \text{centre : } \frac{1}{k}\begin{pmatrix}\sin W\\-\cos W\end{pmatrix}.$$

References

[1] C. de Boor, K. Höllig, M. Sabin, High accuracy geometric Hermite interpolation, Comput. Aided Geom. Design 4 (1987) 269–278.

[2] G. E. Farin, Curves and Surfaces for Computer-Aided Geometric Design: A Practical Guide, 4th Edition, Academic Press, New York, 1997.

[3] K. Höllig, J. Koch, Geometric Hermite interpolation with maximal order and smoothness, Comput. Aided Geom. Design 13 (1996) 681–695.

[4] E.T.Y. Lee, The rational Bézier representation for conics, in: G.E. Farin (Ed.), Geometric Modeling: Algorithms and New Trends, SIAM, Philadelphia, PA, 1987, pp. 3–19.

[5] D.S. Meek, R.S.D. Thomas, Hermite interpolation with a pair of spirals, Comput. Aided Geom. Design 10 (1993) 491–507.

[6] D.S. Meek, D.J. Walton, Clothoid spline transition spirals, Math. Comp. 59 (1992) 117–133.

[7] D.S. Meek, D.J. Walton, Planar G^2 Hermite interpolation with some fair, C-shaped curves, Technical Report Number TR 00/14, Department of Computer Science, University of Manitoba, 2000.

[8] D.S. Meek, D.J. Walton, Planar G^2 Hermite interpolation with a fair, S-shaped curve, J. Comput. Appl. Math. (2002), submitted for publication.

[9] H. Pottmann, Locally controllable conic splines with curvature continuity, ACM Trans. Graphics 10 (1991) 366–377.